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ON SEMIDERIVATIONS OF NEAR-RINGS

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Abstract. In this paper, we introduce the notion of a semiderivation on near-rings, and we try to generalize some properties of prime rings with derivations to prime near-rings with semiderivations.

1. Introduction

J. C. Chang [7] studied on semiderivations of prime rings. He obtained some results of derivations of prime rings into semiderivations. H. E. Bell and W. S. Martindale III [1] investigated the commutativity property of a prime ring by means of semiderivations. C. L. Chuang [8] studied on the structure of semiderivations in prime rings. He obtained some remarkable results in connection with semiderivations. J. Bergen and P. Grzesczuk [3] obtained the commutatity properties of semiprime rings with the help of skew(semi)-derivations. A. Firat [9] generalized some results of prime rings with derivations to the prime rings with semiderivations. In this paper, we introduce the notion of a semiderivation on near-rings, and we try to generalize some properties of prime rings with derivations to prime near-rings with semiderivations.

2. Preliminaries

In this section, we include some elementary aspects that are necessary for this paper.

By a *near-ring* we mean a non-empty set N with two binary operations "$+$" and "$\cdot$" satisfying the following axioms:

(a) $(N, +)$ is a group,

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(b) $(N, \cdot)$ is a semigroup,

(c) $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in N$.

Precisely speaking, it is a left near-ring because it satisfies the left distributive law. We will use the word "near-ring" in stead of "left near-ring". We denote xy instead of $x \cdot y$. Note that $x0 = 0$ and $x(-y) = -xy$ but in general $0x \neq 0$ for some $x \in N$. Let N be a near-ring. Then

N is said to be *prime* if $aNb = 0$ implies $a = 0$, or $b = 0$ for all $a, b \in N$. N is said to be 2-*torsion free* if $2a = 0$ implies $a = 0$ for all $a \in N$.

For any $x, y \in N$, $[x, y], x \circ y$ represent $xy - yx, xy + yx$ respectively. The symbol $Z(N)$ represent the multiplicative center of N, that is, $Z(N) = \{x \in N | xy = yx \text{ for all } y \in N\}$. A mapping $f : N \to N$ is said to be it commuting on N if $[f(x), x] = 0$ for all $x \in N$ and is said to be centralizing on N if $[f(x), x] \in Z(N)$ for all $x \in N$. Let N be a near-ring. An additive mapping $f : N \to N$ is called a *derivation* if

$$f(xy) = f(x)y + xf(y)$$

for all $x, y \in N$.

3. Semiderivations of prime near-rings

In what follows, let N denote a near-ring with center $Z(N)$ unless otherwise specified.

Definition 3.1. Let N be a near-ring. An additive mapping $f : N \to N$ is called a *semiderivation* associated with a surjective function $g : N \to N$ if

(a) $f(xy) = f(x)g(y) + xf(y) = f(x)y + g(x)f(y)$,

(b) $f(g(x)) = g(f(x))$, for all $x, y \in N$.

If $g = I$, i.e., an identity mapping of N, then all semiderivations associated with g are merely ordinary derivations. If g is any endomorphism, then semiderivations are of the form $f(x) = x - g(x)$.

Example 3.2. Let N_1 and N_2 be two semiprime near-rings and $N = N_1 \bigoplus N_2$. If $\alpha_1 : N_1 \to N_1$ be an additive map, $\alpha_2 : N_2 \to N_2$ be a left and right N_2 module map, which is not a derivation. Define a mapping $f : N \to N$ such that $f((r_1, r_2)) = (0, \alpha_2(r_2))$ and $g : N \to N$ such that $g((r_1, r_2)) = (\alpha_1(r_1), 0)$, where $r_1 \in N_1, r_2 \in N_2$. Define addition and multiplication on N by $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$ and $(a_1, b_1)(a_2, b_2) = (a_1a_2, b_1b_2)$ for all $a_1, a_2 \in N_1, b_1, b_2 \in N_2$. Then it can be easily verified that f is a semiderivations on N, with associated map g which is not a derivation.

We begin with the following lemma, which is essential in developing the proof of our main result.

LEMMA 3.3. *Let N be a prime near-ring and $a \in N$. If f is a nonzero semiderivation associated with a surjective function $g : N \to N$. If $af(N) = 0$, then $a = 0$.*

Proof. Suppose that $af(N) = 0$ and $x \in N$. Since $f \neq 0$, there is $y \in N$ such that $f(y) \neq 0$, and

$$af(xy) = 0.$$

Hence we obtain

$$a(f(x)g(y) + xf(y)) = af(x)g(y) + axf(y) = 0. \tag{3.1}$$

Since $af(x) = 0$, we have $axf(y) = 0$ for every $x \in N$. Since N is prime and $f(y) \neq 0$, we get $a = 0$. □

LEMMA 3.4. *Let N be a prime near-ring and f be a semiderivation associated with a surjective function $g : N \to N$. Then*

$$(f(x)g(y) + xf(y))z = f(x)g(y)z + xf(y)z$$

for all $x, y, z \in N$.

Proof. For any $x, y \in N$, we have

$$\begin{aligned} f((xy)z) &= f(xy)g(z) + xyf(z) \\ &= (f(x)g(y) + xf(y))g(z) + xyf(z) \\ &= (f(x)g(y) + xf(y))z + xyf(z) \qquad (g \text{ is onto}) \end{aligned}$$

On the other hand,

$$\begin{aligned} f(x(yz)) &= f(x)g(yz) + xf(yz) \\ &= f(x)g(yz) + xf(y)g(z) + xyf(z) \\ &= f(x)g(y)z + xf(y)z + xyf(z) \qquad (g \text{ is onto }) \end{aligned}$$

Combining both expressions of $f(xyz)$, we have

$$(f(x)g(y) + xf(y))z = f(x)g(y)z + xf(y)z$$

for all $x, y, z \in N$. □

LEMMA 3.5. ([4]) *Let N be a 2-torsion free prime near-ring and let f be a nonzero semiderivation associated with a surjective function $g : N \to N$. If $f(N) \subseteq Z(N)$, then N is commutative.*

PROPOSITION 3.6. *Let N be a prime near-ring. If N admits a nonzero semiderivation f with g such that $f(x \circ y) = x \circ y$ for all $x, y \in N$, then N is commutative.*

Proof. By hypothesis, we have

$$f(x \circ y) = x \circ y, \quad \forall\, x, y \in N. \tag{3.2}$$

Replacing y by xy in (3.2), we get

$$f(x \circ (xy)) = x^2 y + xyx, \quad \forall\, x, y \in N. \tag{3.3}$$

Since $x \circ (xy) = x(x \circ y)$, we obtain

$$f(x \circ (xy)) = f(x)g(x \circ y) + xf(x \circ y), \text{ for all } x, y \in N.$$

By the equation (3.2), we get

$$x \circ (xy) = f(x)(x \circ y) + x(x \circ y), \text{ for all } x, y \in N.$$

Hence $f(x)(x \circ y) = 0$ for all $x, y \in N$. This implies

$$f(x)xy = -f(x)yx, \quad \forall\, x, y \in N. \tag{3.4}$$

Substituting yz for y in (3.4), we obtain for all $x, y \in N$,

$$-f(x)yzx = f(x)xyz = (-f(x)yx)z = f(x)y(-x)z. \tag{3.5}$$

Since $-f(x)yzx = f(x)y(-x)z$, (3.5) becomes

$$f(x)yz(-x) = f(x)y(-x)z, \quad \forall\, x, y \in N. \tag{3.6}$$

Taking $-x$ instead of x in (3.6), we obtain

$$f(-x)yzx = f(-x)yxz, \quad \forall\, x, y \in N. \tag{3.7}$$

Hence $f(-x)y(zx - xz) = 0$ and so

$$f(-x)N[z, x] = 0, \quad \forall\, x, y \in N. \tag{3.8}$$

By primeness, we have either $x \in Z(N)$ or $f(-x) = 0$. That is,

$$f(x) = 0 \text{ or } [x, z] = 0, \quad \forall\, x, y \in N. \tag{3.9}$$

From (3.9), it follows that for each fixed $x \in N$, we have

$$f(x) = 0 \text{ or } x \in Z(N). \tag{3.10}$$

But $x \in Z(N)$ also implies that $f(x) \in Z(N)$, and so the equation (3.8) forces to

$$f(x) \in Z(N), \quad \forall\, x, y \in N. \tag{3.11}$$

In light of (3.11), $f(N) \subset Z(N)$ and using Proposition 3.5, we conclude that N is commutative. $\square$

Theorem 3.7. *Let N be a prime near-ring. If N admits a nonzero semiderivation f with g such that $f(x \circ y) = -(x \circ y)$ for all $x, y \in N$, then N is commutative.*

Proof. By hypothesis,

$$f(x \circ y) = -(x \circ y), \quad \forall\, x, y \in N. \tag{3.12}$$

Replacing y by xy in (3.12), we have, for all $x, y \in N$,

$$\begin{aligned} f(x \circ xy) &= -(x \circ xy) \\ f(x(x \circ y)) &= -x(x \circ y). \end{aligned}$$

Hence we have $f(x)g(x \circ y)) + xf(x \circ y) = -x(x \circ y)$. Using (3.12) in the above equation (3.12) and g is onto, we get

$$f(x)(x \circ y) = 0$$

for all $x, y \in N$. The rest of the proof is as in the proof Theorem 3.6. □

Theorem 3.8. *Let N be a prime near-ring. If N admits a nonzero semiderivation f with g such that $f(x) \circ y = x \circ y$ for all $x, y \in N$, then N is commutative.*

Proof. Suppose that

$$f(x) \circ y = x \circ y, \quad \forall\, x, y \in N. \tag{3.13}$$

Replacing x by xy in (3.13), we get

$$f(xy) \circ y = xy \circ y = (x \circ y)y.$$

Using the equation (3.13), we get $f(xy) \circ y = (f(x) \circ y)y$, and so

$$f(xy)y + yf(xy) = f(x)y^2 + yf(x)y$$

and

$$f(x)g(y)y + xf(y)y + yf(x)g(y) + yxf(y) = f(x)y^2 + yf(x)y.$$

Hence we have

$$f(x)y^2 + xf(y)y + yf(x)y + yxf(y) = f(x)y^2 + yf(x)y$$

since g is onto. Therefore we obtain $xf(y)y + yxf(y) = 0$ for all $x, y \in N$, and so

$$yxf(y) = -xf(y)y, \quad \forall\, x, y \in N. \tag{3.14}$$

Replacing x by xz in the equation (3.14), we get for all $x, y \in N$,

$$\begin{aligned} yxzf(y) &= -xzf(y)y = -x(zf(y)y) \\ &= -x(-yzf(y)) = -x(-y)zf(y). \end{aligned}$$

The last expression reduced to

$$yxzf(y) = -x(-y)zf(y), \quad \forall\ x, y \in N. \tag{3.15}$$

Since $-yxzf(y) = (-y)xzf(y)$, we have

$$(-y)xzf(y) = x(-y)zf(y), \quad \forall\ x, y \in N. \tag{3.16}$$

Taking $-y$ instead of y in (3.16), we get $yxzf(-y) = xyzf(-y)$ for all $x, y \in N$. Hence $(yx - xy)zf(-y) = 0$, and so $[y, x]zf(-y) = 0$, that is,

$$[y, x]Nf(-y) = 0, \quad \forall\ x, y \in N. \tag{3.17}$$

By primeness, we have either $y \in Z(N)$ or $f(-y) = 0$ for all $y \in N$. Accordingly,

$$f(y) = 0 \text{ or } y \in Z(N), \quad \forall\ y \in N. \tag{3.18}$$

But $y \in Z(N)$ also implies that $f(y) \in Z(N)$, and so the equation (3.17) forces to

$$f(y) \in Z(N), \quad \forall\ y \in N. \tag{3.19}$$

In light of (3.19), $f(N) \subset Z(N)$ and using Proposition 3.5, we conclude that N is commutative. □

Theorem 3.9. *Let N be a prime near-ring. If N admits a nonzero semiderivation f with g such that $f(x \circ y) = [x, y]$ for all $x, y \in N$, then N is commutative.*

Proof. Let

$$f(x \circ y) = [x, y], \quad \forall\ x, y \in N. \tag{3.20}$$

Replacing y by yx in (3.20), we obtain

$$f((x \circ y)x) = [x, y]x.$$

Hence we have

$$f(x \circ y)g(x) + (x \circ y)f(x) = [x, y]x.$$

Since g is onto and by (3.20), we get

$$(x \circ y)f(x) = 0, \quad \forall\ x, y \in N. \tag{3.21}$$

Replacing y by zy in (3.21), we obtain $(x(zy) + (zy)x)f(x) = 0$ for all $x, y, z \in N$. Now, application of (3.21) yields $yxf(x) = -xyf(x)$. Combining this fact with the latter relation, we have $xz+z(-x))yf(x) = 0$ for all $x, y, z \in N$. This implies that $[x, z]yf(x) = 0$ for all $x, y, z \in N$. That is,

$$[x, z]Nf(x) = 0, \quad \forall\ x, y \in N. \tag{3.22}$$

Since N is a prime near-ring, for each $x \in N$, we have either $f(x) = 0$ or $x \in Z(N)$ for all $x \in N$. But $x \in Z(N)$ also implies that $f(x) \in Z(N)$, and so the equation (3.22) forces to

$$f(x) \in Z(N), \quad \forall\, x, y \in N. \tag{3.23}$$

In light of (3.23), $f(N) \subset Z(N)$ and using Proposition 3.5, we conclude that N is commutative. □

THEOREM 3.10. *Let N be a prime near-ring. If N admits a nonzero semiderivation f with g such that $f[x, y] = x \circ y$ for all $x, y \in N$, then N is commutative.*

Proof. Let

$$f([x, y]) = x \circ y, \quad \forall\, x, y \in N. \tag{3.24}$$

Replacing y by yx in (3.24), we get, for all $x, y \in N$,

$$f([x, y]x) = (x \circ y)x,$$

and so we obtain

$$f([x, y])g(x) + [x, y]f(x) = (x \circ y)x.$$

Since g is onto and using the equation (3.24), we get

$$[x, y]f(x) = 0, \tag{3.25}$$

Replacing y by yz in (3.25), we obtain

$$[x, y]Nf(x) = 0, \quad \forall\, x, y \in N. \tag{3.26}$$

Since N is a prime near-ring, for each $x \in N$, we have either $f(x) = 0$ or $x \in Z(N)$ for all $x \in N$. But $x \in Z(N)$ also implies that $f(x) \in Z(N)$, and so the equation (3.26) forces to

$$f(x) \in Z(N), \quad \forall\, x, y \in N. \tag{3.27}$$

In light of (3.27), $f(N) \subset Z(N)$ and using Proposition 3.5, we conclude that N is commutative. □

PROPOSITION 3.11. *Let N be a 2-torsion free prime near-ring and let f is a semiderivation associated with a surjective function $g : N \to N$. If $f^2(x) = 0$ for all $x \in N$, then $f = 0$.*

Proof. By hypothesis, we have for all $x \in N$,

$$f^2(x) = 0.$$

Replacing x by xy in the above equation, we obtain $f^2(xy) = 0$ for all $x, y \in N$. Hence, for any $x, y \in N$,

$$\begin{aligned} 0 &= f(f(xy)) \\ &= f(f(x)g(y) + xf(y)) \\ &= f^2(x)g(g(y)) + f(x)f(g(y)) + f(x)g(f(y)) + xf^2(y) \\ &= 2f(x)f(g(y)). \end{aligned}$$

Since N is 2-torsion free and g is surjective, we have $f(x)f(y) = 0$ for all $x, y \in N$. Replacing y by yz, we get for all $x, y, z \in N$,

$$\begin{aligned} 0 &= f(x)f(yz) \\ &= f(x)f(y)g(z) + f(x)yf(z) \\ &= f(x)yf(z). \end{aligned}$$

Hence we obtain $f(x)sf(z) = 0$ for all $x, z \in N$. since N is prime, $f(x) = 0$ or $f(z) = 0$ for all $x, z \in N$. That is, in both cases, $f = 0$. $\square$

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